Published by IOP Publishing for SISSA

Received: *November 18, 2008*
Accepted: *December 8, 2008*
Published: *December 29, 2008*

UV-divergences of Wilson loops for gauge/gravity duality

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Abstract: We analyze the structure of the UV divergences of the Wilson loop for a general gauge/gravity duality. We find that, due to the presence of a nontrivial NSNS B-field and metric, new divergences that cannot be subtracted out by the conventional Legendre transform may arise. We also derive conditions on the B-field and the metric, which when satisfied, the leading UV divergence will become linear, and can be cancelled out by choosing the boundary condition of the string appropriately. Our results, together with the recent result of arXiv:0807.5127, where the effect of a nontrivial dilaton on the structure of UV divergences in Wilson loop is analysed, allow us to conclude that Legendre transform is at best capable of cancelling the linear UV divergences arising from the area of the worldsheet, but is incapable to handle the divergences associated with the dilaton or the B-field in general. We also solve the conditions for the cancellation of the leading linear divergences generally and find that many well-known supergravity backgrounds are of these kinds, including examples such as the Sakai-Sugimoto QCD model or $\mathcal{N} = 1$ duality with Sasaki-Einstein spaces. We also point out that Wilson loop in the Klebanov-Strassler background have a divergence associated with the B-field which cannot be cancelled away with the Legendre transform. Finally we end with some comments on the form of the Wilson loop operator in the ABJM superconformal Chern-Simons theory.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence states the equivalence of string theory on $AdS_5 \times S^5$ to the $\mathcal{N} = 4$ supersymmetric Yang-Mills [1−4]. According to this correspondence, there exists a map between gauge invariant operators in the field theory and states in the string theory. The correspondence is well understood for the case of half BPS local operators where the dual string states are D-branes in the bulk [5, 6]. Wilson loop operator is another class of gauge invariant operator. In the limit of $N \to \infty$ and large $\lambda = g^2 N \gg 1$, the expectation value of a special class of Wilson loops in the $\mathcal{N} = 4$ SYM theory can be computed using the supergravity dual picture in terms of a dual string worldsheet [7−9]. These Wilson loop operator takes the form [7]

$$W[C] = \frac{1}{N} \mathrm{Tr}\, P \exp \left(\oint_C d\tau (i A_\mu \dot{x}^\mu + \varphi_i \dot{y}^i) \right), \tag{1.1}$$

where the trace is over the fundamental representation of the gauge group G, A_μ are the gauge fields and φ_i are the six real scalars. The loop C is parametrized by the variables $(x^\mu(\tau), y^i(\tau))$, where $(x^\mu(\tau))$ determines the actual loop in four dimensions, and $(y^i(\tau))$ parametrizes the coupling to the scalars. Moreover the condition

$$\dot{x}^2 = \dot{y}^2 \tag{1.2}$$

is satisfied. The expectation value is given in terms of supergravity as

$$\langle W[C] \rangle = B e^{-\sqrt{\lambda} \tilde{I}}, \tag{1.3}$$

where the prefactor B has a dependence on the loop C which is subleading for large λ and $\tilde{I}$ is the Legendre transform of the worldsheet action I with respect to some of the loop variables [9]. The Legendre transform is needed because some of the worldsheet scalars satisfy Neumann boundary conditions instead of Dirichlet boundary conditions. The area I has a linear UV divergence $1/\epsilon$ since the metric has a scale factor which diverges as one goes near the boundary. It was demonstrated that [9] the application of the Legendre transform removes this UV divergence from the area and the result $\tilde{I}$ is finite.

So far there has not been much discussions on the structure of the UV divergences and their cancellation for Wilson loops in more general gauge/gravity correspondence beyond the original $AdS_5 \times S^5$ case. In a general supergravity background where the metric is different from the simple $AdS_5 \times S^5$ one, and where a nontrivial B-field and dilaton could be present, there can be new kind of UV divergences. It is interesting to ask whether the implementation of the Legendre transform can cure all the UV divergences or not. In [10], the effects of a varying dilaton were analysed by including the Fradkin-Tseytlin term for the dilaton [11]. It was found that new UV-divergent terms proportional to $\sqrt{1/\epsilon}$ and $\log 1/\epsilon$ occurs.[1] Moreover these divergent terms cannot be subtracted away by the application of Legendre transform. A direct subtraction is applied to extract a finite result. However, the subtraction of the log-divergent term is associated with a finite ambiguity and further physical input is needed to fix the supergravity prediction for the expectation value of the Wilson loop. This is unlike the cancellation of the leading linear divergence in the Polyakov action through a quadratic constraint on the loop variables, which has a nice geometrical and physical interpretation.

In this paper, we focus on the gravity dual analysis of the UV divergences from a nontrivial metric and B-field. The main motivation of our work is to provide a general analysis of the kind of UV divergence that may occur in the Wilson loop correspondence and to provide a prescription for their cancellation. We show indeed in general there are new kinds of UV divergences associated with the metric and the B-field that cannot be cancelled away by the Legendre transformation. However, when certain asymptotic conditions for the metric and the B-field are satisfied, the leading UV divergence becomes linear and one can cancel out the divergence with the Legendre transform by choosing the open string boundary condition appropriately. Things are different for the B-field. We find that the situation is similar to the dilaton: in general the divergences (if any) associated with the B-field cannot be cancelled by the Legendre transformation.

Another motivation of this work is to understand the role of supersymmetry in the holographic correspondence of Wilson loop in a general gauge/gravity duality. In the $\mathcal{N} = 4$ case, the Wilson loop operator (1.1) preserves some amount of local Poincare supersymmetry and is sometimes referred to as "locally BPS". One may wonder if the finiteness of the Wilson loop is related to the preservation of local supersymmetry. Wilson loop operator, being a nonlocal divergent functional, cannot be renormalized by the ordinary R-operation [12] restricted to the local operators. The renormalization properties of Wilson

[1] These divergences were computed for the worldsheet associated with the Wilson line operator with fermion bilinear insertion. However it is easy to see that these divergences are common to Wilson loop too.

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loop with pure glue has been studied in, e.g. [13−15], and it was found that, apart from the conventional wavefunction and coupling renormalization, the only divergence in $W[C]$ is a factor e^{-KL}, where K is a regularization dependent linear divergent constant and L is the length of the loop. This is independent of the form of C and hence the Wilson loop is multiplicative renormalizable. In $\mathcal{N} = 4$ SYM there is no wavefunction renormalization or coupling renormalization, thus the finiteness of the expectation value of the locally BPS Wilson loop means that the multiplicative renormalization factor is finite. As is common in a supersymmetric field theory, it is natural to associate the absence of renormalization of this class of Wilson loop operators with the presence of local supersymmetry, and to suspect that the later is responsible for it. It is thus interesting to consider Wilson loop which preserves less or no local supersymmetry and check if this is correct.

In the previous paper [16], we started to investigate this question by considering the Wilson loop correspondence in the Lunin-Maldacena duality [17]. The gauge theory is given by a marginal β-deformation of the $\mathcal{N} = 4$ SYM and has $\mathcal{N} = 1$ superconformal symmetries. Configuration of minimal surfaces that are dual to field theory Wilson loop were constructed in [18]. We proposed a form of Wilson loop operator that is the dual of these string configurations. We also found that, although these operators do not preserve any local supersymmetry, they have finite expectation value (both in perturbation theory, which we computed up to order $(g^2N)^2$, and from supergravity). In supergravity, the absence of divergence is due to some special properties satisfied by the metric and the B-field. In field theory, we called these operators "near" local BPS in order to distinguish them from generic non-BPS Wilson loops whose expectation values are infinite Although the operator is non-BPS, still there is the possibility that the cancellation of the UV divergence is due to the underlying $\mathcal{N} = 1$ supersymmetric dynamics.

In this paper, we find that the finiteness of the Wilson loop has nothing to do with supersymmetry at all. As in the $AdS_5 \times S^5$ case, the boundary constraint of the worldsheet has an intermediate interpretation as a constraint on the loop variables of the field theory Wilson loop operator. It is a pure coincidence that this loop constraint also implies a preservation of local Poincare supersymmetry in the $\mathcal{N} = 4$ SYM theory. In general, this condition has nothing to do with preservation of any supersymmetry. In fact, as we will see, the multi-parameters β-deformed supergravity background [19] is an example where the Wilson loop expectation value is finite and where the background is not supersymmetric.

The plan of the paper is as follows. In section 2, we present our analysis of the UV divergence in the supergravity Wilson loop associated with the B-field and the metric. In general the divergence that may arises from the B-field coupling is of a different structure from that in the Legendre transform and so cannot be subtracted away. For background where such divergences are absent, the leading order divergence arises from the area and it can be cancelled away using Legendre transform if certain asymptotic conditions are satisfied for the metric and the B-field and if the boundary coordinate of the open string satisfy a certain constraint. As a consistency check, we show that this loop constraint guarantees that the loop equation is satisfied. Subleading divergences could be present in general. We provide a stronger criteria on the supergravity background where the subleading divergences are absent and the Wilson loop is expected to be finite. In section 3, we analyze the conditions for the cancellation of leading divergence and show that they can be solved quite generally. Some explicit backgrounds which satisfy these conditions are given as examples. Many of them also satisfy the stronger form of the cancellation conditions and so for these backgrounds, Wilson loop computed using the supergravity description (1.3) is finite. As a final example, we consider the Klebanov-Strassler background and show that the leading linear divergence in the area can be cancelled away as usual. However there are subleading divergences of order $(\log \epsilon)^2$ associated with the B-field and this cannot be cancelled away with the Legendre transform. We end with some comments on the form of the Wilson loop operator in the three dimensional $\mathcal{N} = 6$ supersymmetric Chen-Simons theory of Aharony, Bergman, Jafferis and Maldacena [20] (ABJM).

2. Structures of UV divergence in the Wilson loop in general supergravity background

2.1 Conditions on the supergravity background and the string worldsheet for cancellation of leading order divergence

Consider a general supergravity background. The string worldsheet is sensitive to the metric, NSNS B-field and the dilaton. The structure of UV divergence associated with a varying dilaton has been analysed in [10] and we will focus on analysing the effect of a general metric and transverse B field on the UV divergences of the supergravity Wilson loop. Denote the metric in the string frame as

$$ds^2 = G_{\mu\nu} dX^\mu X^\nu + G_{ij} dY^i dY^j, \tag{2.1}$$

where $\mu, \nu = 1, \cdots, m$ denotes the indices of a m-dimensional spacetime; and $i, j = 1, \cdots, n$ denotes the indices of a n-dimensional internal manifold. For this metric to be relevant for a holographic correspondence, we assume that the metric has a (conformal) boundary at $Y = 0$, where $Y := \sqrt{(Y^i)^2}$ is the radial variable and is of length dimension. It is also convenient to introduce the angular variables θ^i where $Y^i = Y\theta^i$ with $\theta^{i2} = 1$. We will assume that in the leading order in Y, the metric have the following asymptotic dependence near the boundary:

$$G_{\mu\nu} = \frac{h_{\mu\nu}}{Y^\alpha} + \cdots, \qquad G_{ij} = \frac{k_{ij}}{Y^\beta} + \cdots, \quad \text{as } Y \to 0 \tag{2.2}$$

for $\alpha, \beta \geq 0$. Here $h_{\mu\nu}, k_{ij}$ are functions of θ^i only and $\cdots$ denotes subleading terms.

Next let us analyze the string boundary condition. Let $(\sigma_1, \sigma_2) = (\tau, \sigma)$ be the worldsheet coordinates. The worldsheet action of the string is

$$I = \int_\Sigma d^2\sigma (\sqrt{\det g} - iB_{ij}\partial_1 Y^i \partial_2 Y^j), \tag{2.3}$$

where $g_{\alpha\beta} = G_{IJ}\partial_\alpha X^I \partial_\beta X^J$ is the induced metric. We note that since the worldsheet is an open one, the B field coupling itself is not invariant under the gauge transformation $\delta B = d\Lambda$. In order to be gauge invariant, the B term should be supplemented with a boundary coupling $\int_{\partial\Sigma}\mathcal{A}$. Without writing this term, we are assuming we are in a gauge where $\mathcal{A}=0$ and B is the corresponding potential in this gauge. However how to fix this choice of B-field is a subtle issue. Similar subtlety also arise in the computation of Wilson loop expectation value using D3-brane dual where one need to know the form of the RR 4-form potential C_4 used in the WZ coupling of the D3-brane [21]. There a symmetry criteria is used to pick a certain natural form of C_4. We will assume that similar considerations can be applied and the correct form of B field is used in the analysis below.

The equation of motion implies the Hamilton-Jacobi equation

$$G^{ij}(P_i - iB_{ik}\partial_1 Y^k)(P_j - iB_{jl}\partial_1 Y^l) + G^{\mu\nu}P_\mu P_\nu = G_{ij}\partial_1 Y^i \partial_1 Y^j + G_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu \quad (2.4)$$

where

$$P_i = G_{ij}J_1{}^\beta \partial_\beta Y^j + iB_{ij}\partial_1 Y^j, \quad P_\mu = G_{\mu\nu}J_1{}^\beta \partial_\beta X^\nu. \quad (2.5)$$

are the momentum and

$$J_\alpha{}^\beta = \frac{1}{\sqrt{g}} g_{\alpha\gamma}\epsilon^{\gamma\beta} \quad (2.6)$$

is the complex structure ($\alpha,\beta=1,2$) on the worldsheet. Substitute the conjugate momentum, we obtain

$$\frac{k_{ij}}{Y^{\beta-\alpha}} J_1{}^\alpha \partial_\alpha Y^i J_1{}^\beta \partial_\beta Y^j + h_{\mu\nu} J_1{}^\alpha \partial_\alpha X^\mu J_1{}^\beta \partial_\beta X^\nu = \frac{k_{ij}}{Y^{\beta-\alpha}} \partial_1 Y^i \partial_1 Y^j + h_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu \quad (2.7)$$

near $Y=0$.

One like to know how this equation put constraint on the boundary variables of the theory. To do this we need the boundary conditions for the string coordinates. Suppose that the Wilson loop is parametrized by $(x^\mu(\sigma_1), y^i(\sigma_1))$ and choose the world-sheet coordinates such that the boundary is located at $\sigma_2 = 0$. First we have the Dirichlet boundary condition for the coordinates

$$X^\mu(\sigma_1, 0) = x^\mu(\sigma_1). \quad (2.8)$$

For the remaining coordinates $Y^i(\sigma_1,\sigma_2)$, due to the presence of the B-field, we propose the mixed boundary condition

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) + iB^k{}_l \partial_1 Y^l(\sigma_1, 0) = E^k{}_l\, \dot{y}^l(\sigma_1), \quad (2.9)$$

where $E^k{}_l$ is some invertible matrix which can depend on Y, θ^i. Its form will be determined later.

For now, focus on the first term on the r.h.s. of (2.7). For a string which terminates at the boundary, it is $Y^i(\sigma_1,0)=0$. This would imply also $\partial_1 Y^i(\sigma_1,0)=0$. If $\beta-\alpha \le 0$, then we can get rid of this term immediately. If $\beta-\alpha>0$, then this term indeterminate. To proceed, we consider a limiting process of letting $Y\to 0$. One can get rid of this term if[2] $\partial_1 Y^i = o(Y^{\frac{\beta-\alpha}{2}})$. As in the $AdS_5\times S^5$ case, the term $h_{\mu\nu}J_1{}^\alpha\partial_\alpha X^\mu J_1{}^\alpha \partial_\alpha X^\nu$ on the

[2]We use the symbol $f=o(g)$ to mean $\lim f/g = 0$, i.e. f tends to infinity slower than g or f tends to zero faster than g. We also use $f=O(g)$ to mean $\lim f/g = k$, $0\le k<\infty$. i.e. f tends to infinity not faster than g or f tends to zero not slower than g or f tends to infinity not faster than g.

l.h.s. of (2.7) has to vanish near a smooth boundary since otherwise the determinant of the induced metric will blow up and this will cost an infinite area. Therefore we arrive at the condition

$$h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{Y^{\beta-\alpha}} k_{ij} J_1{}^\alpha \partial_\alpha Y^i J_1{}^\beta \partial_\beta Y^j \tag{2.10}$$

for a worldsheet which terminates on the boundary $Y = 0$. In order for the condition to make sense, one need $J_1{}^\alpha\partial_\alpha Y^i$ to be of the order of $Y^{\frac{\beta-\alpha}{2}}$.

Before analysing further the boundary condition, let us turn to an analysis of the divergence in the worldsheet action I and its Legendre transform

$$\tilde{I} = I - \oint d\sigma_1 P_i Y^i. \tag{2.11}$$

As in the $AdS_5 \times S^5$ case, the area A may pick up a divergent contribution from the boundary. This can be seen by writing the metric in the form

$$G_{ij} dY^i dY^j = \frac{k_{ij}\theta^i\theta^j}{Y^\beta} dY^2 + \frac{1}{Y^{\beta-2}} k_{ij} d\theta^i d\theta^j + \frac{2}{Y^{\beta-1}} k_{ij}\theta^i d\theta^j dY + \cdots, \tag{2.12}$$

where $\cdots$ denotes terms coming from the subleading expansion terms in the metric (2.2). Near the boundary, A picks up the dominant contribution

$$\int dY\, d\sigma_1 \frac{\sqrt{k_{ij}\theta^i\theta^j}}{Y^{\frac{\alpha+\beta}{2}}} \sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \cdots. \tag{2.13}$$

Since the metric is singular at $Y = 0$, we introduce a regulator $Y = \epsilon$ and evaluate the regularized action for $Y \geq \epsilon$. The divergent part of the area is

$$A = \frac{c}{\epsilon^{(\alpha+\beta)/2-1}} \int d\sigma_1 \sqrt{k_{ij}\theta^i\theta^j} \sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \cdots, \tag{2.14}$$

where $c^{-1} := (\alpha+\beta)/2 - 1$ and $\cdots$ denotes possible subleading divergent terms. The B-field coupling can be written as

$$-i\int B_{ij}\partial_1 Y^i \partial_2 Y^j = -i\int \partial_2(B_{ij}\partial_1 Y^i Y^j) + i\int \partial_2(B_{ij}\partial_1 Y^i) Y^j. \tag{2.15}$$

With the cutoff $Y = \epsilon$, the first term on the r.h.s. contributes the boundary term

$$\oint d\sigma_1 i B_{ij} Y^i \partial_1 Y^j\big|_{Y=\epsilon}, \tag{2.16}$$

which cancels against the B-dependent term from the Legendre transform

$$P_i Y^i = G_{ij} Y^i J_1{}^\alpha \partial_\alpha Y^j + i B_{ij} Y^i \partial_1 Y^j. \tag{2.17}$$

Therefore we can write

$$\tilde{I} = \tilde{I}_A + \tilde{I}_B, \tag{2.18}$$

where

$$\tilde{I}_A := A - \oint d\sigma_1\, G_{ij} Y^i J_1{}^\alpha \partial_\alpha Y^j, \tag{2.19}$$

$$\tilde{I}_B := i \int d^2\sigma\ \partial_2(B_{ij}\partial_1 Y^i)Y^j, \tag{2.20}$$

are the Legendre transform modified contributions of the area and B-coupling term. There is a reason we group the terms in this way. Note that the term $G_{ij}Y^i J_1{}^\alpha\partial_\alpha Y^j$ is of the order of $1/Y^{\frac{\alpha+\beta}{2}-1}$ and is of precisely the same order of divergence as in A. Note also that A has a dependence in $J_1{}^\alpha\partial_\alpha Y^j$ due to (2.10). Thus it is in principle possible to cancel the divergence in A using the term $\oint G_{ij}Y^i J_1{}^\alpha\partial_\alpha Y^j$. On the other hand, the term $\tilde{I}_B$ depends on $\partial_1 Y^i$. This dependence is different from the other terms. Thus the B-field contribution, if divergent, corresponds to a new divergence with a different type of functional dependence on the variables of the theory.

Let us consider a B-field such that

$$B_{ij}\partial_1 Y^i = o\left(\frac{1}{Y^{\frac{\alpha+\beta}{2}}}\right). \tag{2.21}$$

This implies that the divergence in $\tilde{I}_B$ will be subleading compared to $\tilde{I}_A$. This condition also implies that the second term on the l.h.s. of (2.9) behaves asymptotically as

$$iB^k{}_l\partial_1 Y^l = o(Y^{\frac{\beta-\alpha}{2}}). \tag{2.22}$$

Since $J_1{}^\alpha\partial_\alpha Y^k$ is the order of $Y^{\frac{\beta-\alpha}{2}}$, one can drop the B-term in (2.9). It is convenient to define $E^k{}_l = Y^{\frac{\beta-\alpha}{2}}\Lambda^k{}_l$ and the boundary condition (2.9) can be written as

$$J_1^\alpha\partial_\alpha Y^k(\sigma_1, 0) = Y^{\frac{\beta-\alpha}{2}}\Lambda^k{}_l\,\dot{y}^l(\sigma_1). \tag{2.23}$$

The Hamilton-Jacobi equation (2.10) becomes

$$h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = k_{ij}\Lambda^i{}_m\Lambda^j{}_n\dot{y}^m\dot{y}^n. \tag{2.24}$$

This condition will play a key role in the cancellation of the divergences in $\tilde{I}_A$. To see this, note that

$$G_{ij}Y^i J_1{}^\alpha\partial_\alpha Y^j = \frac{1}{Y^{\beta-1}}k_{ij}\theta^i\theta^j J_1{}^\alpha\partial_\alpha Y + \frac{1}{Y^{\beta-2}}k_{ij}J_1{}^\alpha\theta^i\partial_\alpha\theta^j + \cdots, \tag{2.25}$$

where $\cdots$ denotes the subleading contribution from the asymptotic expansion of the metric (2.2). This is to be compared with the leading divergence $\sqrt{k_{ij}\theta^i\theta^j}\cdot\sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}/Y^{\frac{\alpha+\beta}{2}-1}$ in A, which, using (2.10), can be written as follows:

$$\frac{\sqrt{k_{ij}\theta^i\theta^j}}{Y^{\beta-1}}\sqrt{(J_1{}^\alpha\partial_\alpha Y)^2 k_{ij}\theta^i\theta^j + 2YJ_1{}^\alpha\partial_\alpha Y J_1{}^\beta k_{ij}\theta^i\partial_\beta\theta^j + Y^2 J_1{}^\alpha J_1{}^\beta k_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j}. \tag{2.26}$$

Obviously (2.25) and (2.26) cannot match in general. Doing so will require an extra constraint among the derivatives of θ^i and Y, which, first of all, is not obvious it is in consistent with the relation (2.10). Moreover this relation does not have any obvious physical interpretation in field theory. On the other hand there is a particularly simple set of conditions which guarantee that (2.25) and (2.26) are equal, namely,

$$k_{ij}\theta^i = \theta^j, \tag{2.27}$$

$$\beta - \alpha < 2. \tag{2.28}$$

In fact the first condition implies immediately $k_{ij}\theta^i\partial_\alpha\theta^j = 0$ and hence the vanishing of the second term in (2.25) and (2.26); while the second condition says that the last term in (2.26) is subleading compared to the first term. As a result of (2.21), (2.27) and (2.28), we can write

$$G_{ij}Y^iJ_1{}^\alpha\partial_\alpha Y^j = \frac{1}{Y^{\beta-1}}J_1{}^\alpha\partial_\alpha Y + \cdots = \frac{1}{Y^{\beta-1}}\sqrt{k_{ij}J_1{}^\alpha\partial_\alpha Y^iJ_1{}^\alpha\partial_\alpha Y^j} + \cdots \tag{2.29}$$

near $Y = 0$, and the Legendre transform contributes the singular terms

$$\oint d\sigma_1 P_iY^i = \frac{1}{\epsilon^{(\alpha+\beta)/2-1}}\oint d\sigma_1\sqrt{k_{ij}\Lambda^i{}_m\Lambda^j{}_n\,\dot{y}^m\dot{y}^n} + \cdots\,, \tag{2.30}$$

where we have used (2.23). Therefore the leading divergence term in (2.14), (2.30) cancels if $c = 1$, i.e. if the leading divergence is linear:

$$\tilde{I}_A = \frac{1}{\epsilon}\oint\left(\sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - \sqrt{k_{ij}\Lambda^i{}_m\Lambda^j{}_n\,\dot{y}^m\dot{y}^n}\right) + \cdots\,, \tag{2.31}$$

and if the Hamilton-Jacobi condition (2.24) holds. Here $\cdots$ denotes the subleading contribution from the asymptotic expansion of the metric (2.2). Whether there are further subleading singularity (like, for example, $1/\sqrt{\epsilon}$ or $\log\epsilon$ type) or not will depend on the specific details of the asymptotic form of the background metric. Note that since $\partial_1 Y^i$ is of order Y, the sufficient condition (2.21) for the $\tilde{I}_B$-term to be subleading divergent can be written as

$$B_{ij} = o\left(\frac{1}{Y^{\frac{\alpha+\beta}{2}+1}}\right). \tag{2.32}$$

On the other hand, if

$$B_{ij} = o\left(\frac{1}{Y^2}\right), \tag{2.33}$$

then the $\tilde{I}_B$-term is non-divergent.

Summarizing in a general supergravity background, the B-field coupling in the world-sheet action generically generates a divergence which cannot be cancelled with the Legendre transform. A sufficient condition for the B-field contribution to be finite is (2.33). When there is no such divergence, the leading order divergence in the Wilson loop arises from the area and it can be cancelled with the application of Legendre transform if the following conditions are satisfied:

1. supergravity background:

   - The supergravity metric takes the asymptotic form (2.2) near the boundary. Moreover

   $$\alpha + \beta = 4, \quad \beta - \alpha < 2. \tag{2.34}$$

   - The boundary metric $h_{\mu\nu}$ is independent of θ^i. The transverse part of the metric satisfies the boundary condition

   $$k_{ij}\theta^i = \theta^j. \tag{2.35}$$

These conditions are conditions on the background and do not impose any extra constraint on the form of the Wilson loop variables.

2. string worldsheet:

   The boundary constraint (2.24) for the string worldsheet is satisfied.

In general, once the leading UV divergences are cancelled, there may be further subleading singularity (like, for example, $1/\sqrt{\epsilon}$ or $\log \epsilon$ type). An extensive analysis of them will need information on the specific details of the asymptotic form of the background metric, the B-field and the dilaton. Generally we don't expect the subleading divergences can be cancelled with the application of Legendre transform.

A special situation with no further subleading divergence is if the leading correction term in the asymptotic conditions (2.2) and (2.33) are of at least order Y. We will examine some examples of this kind later.

2.2 Comments: boundary constraint as loop constraint

Just as in the original $AdS_5 \times S^5$ case, one would like to interpret the boundary constraint (2.24) for the open string as a condition in the field theory. Since the Wilson loop is specified by the loop variables $\dot{x}^\mu$ and $\dot{y}^i$, and θ^i does not play any role, the loop constraint should not depend on θ^i. This means $h_{\mu\nu}$ should be independent of θ^i. For the same reason, one should choose $\Lambda^k{}_m$ such that $k_{kl}\Lambda^k{}_m\Lambda^l{}_n$ is independent of θ^i. Generally this can be achieved by taking $\Lambda^k{}_m$ of the form

$$\Lambda^k{}_m = \hat{\Lambda}^k{}_l M^l{}_m, \tag{2.36}$$

where $\hat{\Lambda}^k{}_l$ is the vielbein of the metric k_{kl} and $M^l{}_m$ is an invertible matrix which is independent of θ^i but can depends arbitrarily on parameters which have meaning both in supergravity and in the field theory (e.g. the 't Hooft coupling or parameters in the theory such as the β-deformation parameter in the Maldacena-Lunin duality). As a result, the condition (2.24) takes the form

$$h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = a_{ij}\dot{y}^i\dot{y}^j, \qquad i,j = 1,\cdots,n \tag{2.37}$$

where we have defined $a_{ij} := M^n{}_i M^n{}_j$. In general the form of the matrix a_{ij} will be a function of the couplings of the theory and cannot be fixed from the supergravity analysis alone. In the original $\mathcal{N}=4$ SYM case [9], the matrix a_{ij} is given by $a_{ij} = \delta_{ij}$. We have also computed the constraint for the $\mathcal{N}=1$ β-deformed superconformal field theory and find $a_{ij} = \delta_{ij}$ up to λ^2 order in perturbation theory [16]. We emphasize that in general the constraint (2.37) has nothing to do with preservation of any supersymmetry. It is a pure coincidence that this loop constraint also implies a preservation of local Poincare supersymmetry in the $\mathcal{N}=4$ SYM theory.

Let us make a consistency check on the boundary constraint (2.37). In the large N limit of gauge theory, Wilson loop satisfies a closed set of equations called the loop equation [22]. To further justify the supergravity procedure for the computation of the

Wilson loop expectation value, one should check that the supergravity ansatz(1.3) satisfies the loop equation [22]. As in the $AdS_5 \times S^5$ case, although the leading linear divergence cancels out when the loop constraint (2.37) is satisfied, the loop variation does not commute with the constraint and so the linear divergence may gives a divergent contribution and violate the loop equation. We show this is not the case.

The loop derivative operator is given by

$$\hat{L} = \lim_{\eta\to 0} \oint ds \int_{s-\eta}^{s+\eta} ds' \left(\frac{\delta^2}{\delta x^\mu(s')\delta x_\mu(s)} - a^{ij} \frac{\delta^2}{\delta y^i(s')\delta y^j(s)} \right). \qquad (2.38)$$

That this definition is correct can be confirmed by checking that $\hat{L}\langle W\rangle = 0$ in field theory for the Wilson loop operator (1.1). As usual the loop regulator η has to be taken much smaller than the UV cutoff scale ϵ in order to extract the equation of motion terms. Now acting on the supergravity ansatz (1.3) with the the loop operator, we get the leading term in large λ,

$$\lambda \lim_{\eta\to 0} \oint ds \int_{s-\eta}^{s+\eta} ds' \left(\frac{\delta \tilde{I}_A}{\delta x^\mu(s')} \frac{\delta \tilde{I}_A}{\delta x_\mu(s)} - \frac{\delta \tilde{I}_A}{\delta y^i(s')} \frac{\delta \tilde{I}_A}{\delta y_i(s)} \right). \qquad (2.39)$$

Let us now extract the divergent contribution from $\tilde{I}_A$ in (2.31). Given the condition (2.37), we can choose a parametrization such that $h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = a_{ij}\dot{y}^i\dot{y}^j = 1$ and get

$$\hat{L}\langle W\rangle = \frac{\lambda\eta}{\epsilon^2} \oint ds \left(h_{\mu\nu}\ddot{x}_\mu\ddot{x}^\nu - a_{ij}\ddot{y}^i\ddot{y}^j) \right). \qquad (2.40)$$

For a smooth loop the terms in the integral are finite. Therefore by taking η going to zero faster than ϵ^2, we find

$$\hat{L}\langle W\rangle = 0 \qquad (2.41)$$

and the loop equation is satisfied.

3. General solution to the conditions on SUGRA background and examples

3.1 General solution to the metric condition

The condition (2.27) on the metric may look a little restrictive at first sight. We show now that it is in fact satisfied by a general class of metric of the form

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + F(Y)dY^2 + g_{ij}d\theta^i d\theta^j, \qquad (3.1)$$

where θ^i, $i, j = 1, \cdots, n$ are the coordinates of the $n-1$ dimensional space X_{n-1}; and the metric g_{ij} is a function of Y^i, e.g. as in the Klebanov-Strassler metric [23]. The metric can be thought as a warped product of the boundary spacetime $(T, \vec{X})$ and the transverse space (Y, θ^i).

Defining $Y^i = Y\theta^i$ and making the coordinate transformation we get

$$g_{ij}d\theta^i d\theta^j = \frac{1}{Y^2}(g_{kl} + g_{ij}\theta^i\theta^j\theta^k\theta^l - g_{il}\theta^i\theta^k - g_{ki}\theta^i\theta^l)dY^l dY^k. \qquad (3.2)$$

So our metric become

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + G_{ij}dY^i dY^j, \tag{3.3}$$

where

$$G_{ij} := F(Y)\theta^i\theta^j + \frac{1}{Y^2}A_{ij}, \tag{3.4}$$

and

$$A_{ij} := g_{ij} + g_{kl}\theta^k\theta^l\theta^i\theta^j - g_{il}\theta^l\theta^j - g_{jl}\theta^l\theta^i. \tag{3.5}$$

The matrix A_{ij} satisfies the following identity,

$$A_{ij}\theta^j = 0, \tag{3.6}$$

and so

$$G_{ij}Y^j = F(Y)Y^i \tag{3.7}$$

Note that (3.7) is of the form of (2.27). Therefore if F behaves as

$$F(Y) = \frac{1}{Y^\beta}, \quad Y \to 0, \tag{3.8}$$

near the boundary, then the condition (2.27) is satisfied. Therefore if also $\alpha + \beta = 4$ and $\beta - \alpha < 2$, then the metric conditions are satisfied.

It is easy to give example where the condition (2.27) is not satisfied. For example, if we have started with a metric with an additional cross-terms $dYd\theta^i$

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + F(Y)dY^2 + K_i(Y)dYd\theta^i + g_{ij}d\theta^i d\theta^j, \tag{3.9}$$

then under the same coordinate transformation, the additional term takes the form

$$K_i(Y)dYd\theta^i = \frac{1}{Y}\Big(\frac{1}{2}(\theta^k K_l + \theta^l K_k) - (K_i\theta^i)\theta^k\theta^l\Big)dY^k dY^l := \frac{1}{Y}\xi_{kl}dY^k dY^l. \tag{3.10}$$

ξ_{kl} satisfies the following identities

$$\xi_{ij}\theta^j = \frac{1}{2}(K_i - (K_l\theta^l)\theta^i), \qquad \xi_{ij}\theta^i\theta^j = 0, \qquad \xi_{ij}\theta^i\partial\theta^j = \frac{1}{2}K_l\partial\theta^l \tag{3.11}$$

Denote the whole metric as $G_{ij} := H_{ij} + Y^{-1}\xi_{ij}$, where H_{ij} is given by the r.h.s. of (3.4). It is

$$G_{ij}\theta^j = F(Y)\theta^i + \frac{1}{2Y}(K_i - (K_l\theta^l)\theta^i). \tag{3.12}$$

Since the right hand side is generally not proportional to θ^i, the condition (2.27) is no longer satisfied. Note that the cross-terms in (3.9) may be eliminated with a shift of $\theta^i \to \theta^i + a_i(Y)$. However the new θ's will not satisfy the condition $(\theta^i)^2 = 1$ anymore. This is another way to see that the metric conditions are not satisfied.

3.2 Examples

Here we examine some backgrounds with known dual field theories, to which our analysis can be applied.

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Background with $AdS_5 \times X^5$ metric. This is a standard example. The metric of the space can be written as

$$ds^2 = U^2 \sum_{\mu=0}^{3} dX^\mu dX^\mu + \frac{dU^2}{U^2} + dX_5^2. \tag{3.13}$$

where X^5 is an internal compact space. In this case $\alpha = 2 = \beta$ and the condition (2.34) is satisfied. The linear divergence in A is cancelled by the Legendre transform and $\tilde{I}_A$ is finite. Some explicit examples are, $X^5 = S^5$, $\tilde{S}^5$, $\tilde{S}^5_{\gamma_1,\gamma_2,\gamma_3}$, $T^{1,1}$, $Y^{p,q}$, $L^{p,q,r}$, etc., where respectively these spaces are the 5-sphere for the original Maldacena AdS/CFT correspondence [7], the β-deformed 5-sphere for the Lunin-Maldacena β-deformation [17], the multi-parameter β-deformed sphere [19], and the Sasaki-Einstein spaces [24, 25]. The boundary condition for the string minimal surface is

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = \hat{\Lambda}^k{}_m M^m{}_l\, \dot{y}^l(\sigma_1). \tag{3.14}$$

It is easy to see that $\tilde{I}_B$ is finite for these cases. In the $AdS_5 \times S^5$ case or in the duality with Sasaki-Einstein spaces, there is simply no B-field. In the β-deformation or the multi-parameters β-deformation, the B-field is of the form

$$B = \frac{1}{2} B_{ab} d\phi^a d\phi^b, \tag{3.15}$$

where $\sum(\mu_a)^2 = 1$, ϕ^a $(a = 1, 2, 3)$ are the azimuth angles defined by

$$\begin{aligned} Y^1 &= Y\theta^1 = Y\mu_1 \cos\phi_1, & Y^4 &= Y\theta^4 = Y\mu_1 \sin\phi_1, \\ Y^2 &= Y\theta^2 = Y\mu_2 \cos\phi_2, & Y^5 &= Y\theta^5 = Y\mu_2 \sin\phi_2, \\ Y^3 &= Y\theta^3 = Y\mu_3 \cos\phi_3, & Y^6 &= Y\theta^6 = Y\mu_3 \sin\phi_3 \end{aligned} \tag{3.16}$$

and B_{ab} is a function of μ_a. This form of the B-field respects the symmetries of the β-deformed sphere and we will take it to be the B-field where the string is coupled to. In general one may get a different answer by using a different gauge equivalent B-field. This is similar to the situation discussed in [21] where an open D3-brane is employed to compute the expectation value of Wilson loop in higher representation. There the answer is shown to depend on the gauge choice of the RR 4-form potential C_4 which appears in the Wess-Zumino couping. A symmetry argument was used to suggest the natural form of the C_4 to be used.

Obviously the B-term in the worldsheet action is finite. For the piece $B_{ij}Y^i\partial_1 Y^j$ in the Legendre transform, since B_{ij} is of order $1/Y^2$, this term is potentially linear divergent. However this does not happen since, as we have shown in [16], a B-field of the form (3.15) satisfies the condition

$$B_{ij} Y^i = 0 \tag{3.17}$$

exactly. This can be seen easily by noticing that

$$d\phi^1 d\phi^2 = \frac{1}{\mu_1^2\mu_2^2 Y^4}(Y_4Y_5 dY_1\wedge dY_2 + Y_1Y_2 dY_4\wedge dY_5 + Y_1Y_5 dY_2\wedge dY_4 - Y_2Y_4 dY_1\wedge dY_5),$$
$$d\phi^1 d\phi^3 = \frac{1}{\mu_1^2\mu_3^2 Y^4}(Y_4Y_6 dY_1\wedge dY_3 + Y_1Y_3 dY_4\wedge dY_6 + Y_1Y_6 dY_3\wedge dY_4 - Y_3Y_4 dY_1\wedge dY_6),$$
$$d\phi^2 d\phi^3 = \frac{1}{\mu_2^2\mu_3^2 Y^4}(Y_5Y_6 dY_2\wedge dY_3 + Y_2Y_3 dY_5\wedge dY_6 + Y_2Y_6 dY_3\wedge dY_5 - Y_3Y_5 dY_2\wedge dY_6).$$

As a result, the piece $B_{ij}Y^i\partial_1 Y^j$ in the Legendre transform is zero. Therefore, there is no divergence associated with the B-field. This can also be checked using (2.20). For example the contributions from B_{12}, B_{15} to $\partial_2(B_{ij}\partial_1 Y^i)Y^j$ is of the form $\sim \frac{Y_4(Y_2)^2}{Y^4}\partial_1 Y_1 \partial_2 \frac{Y_5}{Y_2}$. This is finite as $Y\to 0$ and so $\tilde{I}_B$ is free from any divergence. Also since there is no subleading correction terms to the metric and the B-field, there is no subleading divergence at all. The Wilson loop is finite.

We remark that the background $AdS_5 \times \tilde{S}^5_{\gamma_1,\gamma_2,\gamma_3}$ for the multi-parameters β-deformation is not supersymmetric, but the Wilson loop expectation value is finite. This clearly shows that supersymmetry or the satisfaction of the BPS condition for the loop is not what is required for the finiteness of Wilson loop expectation value.

Supergravity background with asymptotically $AdS_5\times X^5$ metric. The first kind of example is given by a finite temperature deformation of any of the metric above. For example for $\mathcal{N}=4$ at finite temperature, the metric is

$$ds^2 = U^2\left(-\left(1-\frac{U_T^4}{U^4}\right)dt^2 + (dX^i)^2\right) + \left(1-\frac{U_T^4}{U^4}\right)^{-1}\frac{dU^2}{U^2} + d\Omega_5^2 \tag{3.18}$$

Asymptotically, the metric behaves identically to that of the $AdS_5\times S^5$ background. So the cancellation of the infinity occurs with the same boundary conditions as in the $AdS_5\times S^5$ case. Putting a finite temperature deforms the asymptotic form of the metric with power-like terms and this does not introduce any additional subleading singularity.

Sakai-Sugimoto QCD model. The background consists of a dilaton, a RR 3-form potential and the metric [26]

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{3/2}(\eta_{\mu\nu}dX^\mu dX^\nu + f(U)dz^2) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\ e^\phi &= g_s\left(\frac{U}{R}\right)^{3/4}, \\ f(U) &= 1 - \frac{U_{\rm KK}^3}{U^3}. \end{aligned} \tag{3.19}$$

Here X^μ ($\mu=0,1,2,3$) is the spacetime. $z=X^5$ is periodic and describes the compact direction of the D4-brane. $U>U_{\rm KK}$ corresponds to the radial direction transverse to the D4-brane. With the coordinate transformation $Y=R^2/U$, the metric near the boundary $U=\infty$ reads

$$ds^2 = \left(\frac{R}{Y}\right)^{3/2}(\eta_{\mu\nu}dX^\mu dX^\nu + dz^2) + \left(\frac{R}{Y}\right)^{5/2}(dY^2 + Y^2 d\Omega_4^2). \tag{3.20}$$

In this case $\alpha = 3/2, \beta = 5/2$ and the condition (2.34) is satisfied. The leading UV divergence is a linear one and it can be cancelled with a choice of the boundary condition for the string minimal surface

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = Y^{1/2} M^k{}_l \, \dot{y}^l(\sigma_1). \tag{3.21}$$

The vielbein is trivial since $k_{ij} = \delta_{ij}$ $(i, j = 1, \cdots, 5)$ for the boundary metric. Including the contribution of the pion field φ_0, we propose the following form of the Wilson loop operator for the Sakai-Sugimoto QCD model,

$$W[C] = \frac{1}{N} \mathrm{Tr}\, P \, \exp\left(\oint_C d\tau (i A_\mu \dot{x}^\mu + i\varphi_0 \dot{z} + \varphi_i \dot{y}^i) \right), \tag{3.22}$$

and the constraint is

$$\dot{x_\mu}^2 = \dot{y_i}^2 - \dot{z}^2. \tag{3.23}$$

Moreover since the subleading correction terms to the metric is power-like, therefore there is no further subleading UV divergences.

Klebanov-Strassler background. Another example is the Klebanov-Strassler background [23] which describes a warped deformed conifold. In this case the asymptotic behavior of the metric is different from the power ansatz (2.2). However it is not difficult to repeat our analysis above.

The background has a constant dilaton, a RR 2-form, and the metric and B-field

$$\begin{aligned} ds^2 &= h^{-1/2} m^2 dx_m dx_m + h^{1/2} \frac{3^{1/3}}{2^{4/3}} K \left[\frac{1}{3K^3}(d\tau^2 + (g_5)^2) + \cosh^2 \frac{\tau}{2} [(g_3)^2 + (g_4)^2] \right. \\ &\qquad \left. + \sinh^2 \frac{\tau}{2} [(g_1)^2 + (g_2)^2] \right], \end{aligned} \tag{3.24}$$

$$B = \frac{g_s M}{2} \left[f g_1 \wedge g_2 + k g_3 \wedge g_4 \right], \tag{3.25}$$

where g_i is a basis of invariant one-form on $T^{1,1}$

$$\begin{aligned} g_1 &= \frac{1}{\sqrt{2}}(-s_1 d\phi_1 - c_\psi s_2 d\phi_2 + s_\psi d\theta_2), \qquad & g_2 &= \frac{1}{\sqrt{2}}(d\theta_1 - s_\psi s_2 d\phi_2 - c_\psi d\theta_2), \\ g_3 &= \frac{1}{\sqrt{2}}(-s_1 d\phi_1 + c_\psi s_2 d\phi_2 - s_\psi d\theta_2), \qquad & g_4 &= \frac{1}{\sqrt{2}}(d\theta_1 + s_\psi s_2 d\phi_2 + c_\psi d\theta_2), \\ g_5 &= d\psi + c_1 d\phi_1 + c_2 d\phi_2. \end{aligned} \tag{3.26}$$

The B-field respects the symmetries of $T^{1,1}$ and we will assume that this is the proper B-field where the string is coupled to. h, K, f and k are some functions of τ whose form can be found in [23]. For our purpose, we record their asymptotic form for large τ,

$$\begin{aligned} h &= e^{-\frac{4\tau}{3}}(4\tau - 1) + O(\tau^2 e^{-\frac{10\tau}{3}}), \qquad & K &= 2^{1/3} e^{-\tau/3}\left(1 - \frac{4\tau}{3} e^{-2\tau}\right) + O(e^{-\frac{2\tau}{3}}), \\ f &\to \frac{\tau - 1}{2} - \tau e^{-\tau} + O(\tau e^{-2\tau}), \qquad & k &\to \frac{\tau - 1}{2} + \tau e^{-\tau} + O(\tau e^{-2\tau}). \end{aligned} \tag{3.27}$$

In this limit, the metric becomes

$$ds^2 = h^{-1/2}(r)dx^2 + h^{1/2}(r)ds_6^2, \tag{3.28}$$

where the radial variable is defined by

$$r^3 = r_s^3 e^\tau \tag{3.29}$$

for some resolved scale r_s. The warp factor is

$$h = \frac{1}{r^4}\left(\log\frac{r}{r_s} - \frac{1}{4}\right) + o\left(\frac{1}{r^{10}}\left(\log\frac{r}{r_s}\right)^2\right) \tag{3.30}$$

and ds_6^2 is the cone metric over $T^{1,1}$

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2. \tag{3.31}$$

The B-field behaves

$$B = O\left(\log\frac{r}{r_s}\right)(s_1 d\theta_1 d\phi_1 - s_2 d\theta_2 d\phi_2). \tag{3.32}$$

Putting $Y = 1/r$, we have near the boundary $Y = 0$

$$G_{\mu\nu} = \frac{h_{\mu\nu}}{Y^2\sqrt{\log Y}}\left(1 + O\left(\frac{1}{\log Y}\right)\right), \tag{3.33}$$

$$G_{ij} = k_{ij}\frac{\sqrt{\log Y}}{Y^2}\left(1 + O\left(\frac{1}{\log Y}\right)\right), \tag{3.34}$$

and

$$B_{ij} = O\left(\frac{\log Y}{Y^2}\right). \tag{3.35}$$

Here $h_{\mu\nu} = \eta_{\mu\nu}$ and k_{ij} can be worked out using the metric of $T^{1,1}$. These details will not be important for us. Note that the metric (3.24) is of the form (3.1) and so it satisfies the condition (3.7).

The Hamilton-Jacobi equation (2.7) is replaced by

$$(\log Y)k_{ij}{J_1}^\alpha\partial_\alpha Y^i {J_1}^\beta\partial_\beta Y^j + h_{\mu\nu}{J_1}^\alpha\partial_\alpha X^\mu {J_1}^\beta\partial_\beta X^\nu = (\log Y)k_{ij}\partial_1 Y^i\partial_1 Y^j + h_{\mu\nu}\partial_1 X^\mu\partial_1 X^\nu. \tag{3.36}$$

The string boundary condition is given by the same Dirichlet condition (2.8) and mixed boundary condition (2.9). For a string terminating on the boundary, we have $Y^i(\sigma_1, 0) = 0$. To get rid of the first term on the r.h.s. of (3.36), we require that $\partial_1 Y^i(\sigma_1, 0) = o(1/\sqrt{\log Y})$. This also implies that the B-term in the mixed boundary condition

$$iB^k{}_l\partial_1 Y^l = o(1). \tag{3.37}$$

The Hamilton-Jacobi equation in the limit $Y \to 0$ makes sense if ${J_1}^\alpha\partial_\alpha Y^i(\sigma_1, 0)$ is of the order of $1/\sqrt{\log Y}$. Therefore, we can drop the B-term in the mixed boundary condition (2.9) and write

$${J_1}^\alpha\partial_\alpha Y^i(\sigma_1, 0) = \frac{1}{\sqrt{\log Y}}\Lambda^i{}_j\dot{y}^j(\sigma_1). \tag{3.38}$$

The Hamilton-Jacobi equation finally gives

$$h_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = k_{ij}\Lambda^{i}{}_{m}\Lambda^{j}{}_{n}\dot{y}^{m}\dot{y}^{n}. \tag{3.39}$$

Now we examine the structure of UV divergences. For the area part, it is easy to see that we get the same linear divergence (2.31) as before and so $\tilde{I}_A$ is finite if the loop condition (3.39) is satisfied. As for the B-field, since $\partial_2(B_{ij}\partial_1 Y^i)Y^j$ is of the order of $\log Y/Y$, therefore

$$\tilde{I}_B \sim (\log \epsilon)^2. \tag{3.40}$$

This is a new divergence which can not be cancelled with the Legendre transform.

4. Discussions

In this paper, we have analysed of the structure of UV divergences in the Wilson loop from the supergravity point of view by including the effect of a non-trivial metric and a NSNS B-field. We find that in general there can be new divergences which cannot be cancelled with the Legendre transform. We also find that when certain conditions are satisfied by the B-field and the metric, the leading UV divergence becomes a linear one and this can be cancelled away by choosing the boundary condition of the string appropriately. In general there may still be divergences associated with the B-field, and if they do exist, there is no way to cancel them with the Legendre transform. This is similar to the result of [10] which analysis the effect of a nontrivial dilaton on the structure of UV divergences in Wilson loop. We conclude that Legendre transform is at best capable of cancelling only linear UV divergences, but is incapable to cancelling any subleading divergences which may be present, no matter whether it is due to the dilaton or the NSNS B-field.

We have been concentrating on the structure of UV divergences associated with the string minimal surface. For Wilson loop in higher representations, a more suitable dual description is in terms of D3-branes or D5-branes [21, 27 – 33]. Presumably the correspondence will continue to hold for a more general class of gauge/gravity duality. It will be interesting to analyze the structure of the UV divergences there and to derive the corresponding boundary conditions for the corresponding D-brane description.

Our analysis is performed on the supergravity side. It is an interesting question to check and confirm the form of the loop constraint (2.37) from the field theory perspective. To do this, one need to know the form of the Wilson loop operator that is dual to the supergravity computation. In the simplest case where the field theory has the same number of (adjoint) massless scalar with the dimension of the internal manifold, the natural candidate for the operator is a direct generalisation of (1.1). However, the field theory may have different number of scalar fields in general. This is the case, for example, in the quiver theories that are dual to backgrounds with Sasaki-Einstein spaces [24, 25]. There the form of the Wilson loop operator is unknown. In this example one may try to exponentiate a product of the bifundamental fields in order to construct the Wilson loop. But since scalar field has dimension one in four dimensions, one needs to compensate the dimension with another dimensional quantity. This is not completely clear what it might be in a conformal theory.

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It will be interesting to analyze this further and to construct the Wilson loop operator for these theories.

Finally we end with some remarks on the form of the Wilson loop operator in the 3-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons theory [20], where recently the correspondence of Wilson loop has been analysed [34 – 37] (see also [38] for related discussions). The ABJM theory has a $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge and opposite levels k and $-k$. The matter fields are bifundamental scalar fields A_1, A_2 in the representation $(\mathbf{N}, \bar{\mathbf{N}})$ and anti-bifundamental fields B_1, B_2 in the representation $(\bar{\mathbf{N}}, \mathbf{N})$ and fermions. On the field theory side, a Wilson loop operator which couples to a certain bilinear combination of the bifundamental fields has been considered

$$W[C] = \frac{1}{N} \mathrm{Tr}\, P \exp \left[\oint_C d\tau \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_I{}^J Y^I Y_J^\dagger \right) \right], \tag{4.1}$$

where $Y^I = (A_1, A_2, \bar{B}_1, \bar{B}_2)$ and the curve C is a straight line or a circle. For the special case where C is spacelike and $M = \mathrm{diag}(1, 1, -1, -1)$, the operator is 1/6 BPS. In this case the UV divergences of this operator cancelled in the perturbation theory. It was also argued [35] that this 1/6 BPS Wilson loop operator describes a string smeared over a CP^1 in CP^3. The smeared string perserves a $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subgroup of the $\mathrm{SU}(4)$ isometry, which is precisely the amount of R-symmetry preserved by the operator (4.1) for this particular choice of M. As a smeared configuration, one would not expect to have a relation like (2.23) to relate the worldsheet boundary conditions with the couplings of the scalar fields in the Wilson loop. In general one may consider localized string in CP^3 and ask how it's boundary condition appears in the Wilson loop. We will consider a natural proposal in the following. However it turns out the correct operator has to be more complicated than this.

To describe the string theory on CP^3 (see for example, [39]), it is convenient to use the complex coordinates w^I

$$\sum_{I=1}^4 w^I \bar{w}^I = 1, \tag{4.2}$$

subjected to the constraint

$$\sum_{I=1}^4 (w^I \partial_\alpha \bar{w}^I - \bar{w}^I \partial_\alpha w^I) = 0, \quad \alpha = 1, 2. \tag{4.3}$$

This construction is a realization of the Hopf fibration since the first constraint describes a S^7 and the second constraint describes a $\mathrm{U}(1)$ symmetry which reduces the embedding to CP^3. Using this description, one can think about the transverse space to the boundary spacetime $\mathbf{R}^3$ as described by the four coordinates $Z^I := Y w^I$ where Y is the radial coordinate of AdS_4. In terms of Z^I, we have $\sum_{I=1}^4 Z^I \bar{Z}^I = Y^2$ and

$$\sum_{I=1}^4 (Z^I \partial_\alpha \bar{Z}^I - \bar{Z}^I \partial_\alpha Z^I) = 0, \quad \alpha = 1, 2. \tag{4.4}$$

The string boundary condition is then given by the three Dirichlet condition for the longitudinal coordinates and the eight Neumann boundary conditions

$$J_1^\alpha \partial_\alpha Z^I(\tau,0) = \dot{z}^I(\tau), \quad I = 1,\cdots,4. \tag{4.5}$$

Note that the boundary condition (4.5) is consistent with the constraint in (4.4) since $Z^I(\tau,0) = 0$. In terms of real coordinates $Z^1 = Y^1 + iY^5, Z^2 = Y^2 + iY^6, Z^3 = Y^3 + iY^7, Z^4 = Y^4 + iY^8$, the embedding reads $\sum_{i=1}^8 (Y^i)^2 = Y^2$ and

$$\sum_{I=1}^4 (Y^I \partial_\alpha Y^{I+4} - Y^{I+4}\partial_\alpha Y^I) = 0. \tag{4.6}$$

The boundary condition reads

$$J_1^\alpha \partial_\alpha Y^i(\tau,0) = \dot{y}^i, \quad i = 1,\cdots,8, \tag{4.7}$$

where $z^1 = y^1 + iy^5, z^2 = y^2 + iy^6, z^3 = y^3 + iy^7, z^4 = y^4 + iy^8$.

To write down the Wilson loop, we note that due to the presence of the product gauge group, there are two independent Wilson loops one can write down. Let us concentrate for the moment on the first $\mathrm{U}(N)$, one can form adjoint fields by multiplying the bi-fundamental fields in a certain order. It is natural to consider

$$W = \frac{1}{N}\mathrm{Tr}P\exp\left(\oint_C d\tau(iA_\mu \dot{x}^\mu + \dot{a}_{ab}A_a\bar{A}_b + \dot{b}_{ab}\bar{B}_a B_b)\right) \tag{4.8}$$

where C is a general spacelike curve. This operator is invariant under arbitrary reparametrization $\tau \to \tilde{\tau}$, including orientation reversing ones. Since scalar fields in three-dimensions is of dimension half, the variables a^{ab} and b^{ab} are of length dimension and therefore it make sense to try to identify them with the boundary variables z^I in (4.5). Since A_a (or B_a) is a doublet of $\mathrm{SU}(2)_1$, $A_a\bar{A}_b$ (or $\bar{B}_a B_b$) contains a singlet and a triplet of $\mathrm{SU}(2)_1$. Our proposal is to identify

$$\dot{a}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{i=1}^4 (\sigma^i)_{ab}\dot{y}^i, \quad \dot{b}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{i=1}^4 (\sigma^i)_{ab}\dot{y}^{i+4} \tag{4.9}$$

where $\sigma^i = (\tau^1,\tau^2,\tau^3,1)$ and $\tau^{1,2,3}$ are the Pauli matrices. Note that the ABJM theory is manifestly invariant under $\mathrm{SU}(2)\times\mathrm{SU}(2)$ of the $\mathrm{SU}(4)$ R-symmetry. Therefore (4.8) respects this symmetry if we assign (y^1,y^2,y^3) (respectively (y^5,y^6,y^7)) to be a triplet and y^4 (respectively y^8) to be a singlet $\mathrm{SU}(2)_1$ (respectively $\mathrm{SU}(2)_2$). For convenience, we have put a factor of $2\sqrt{2}\pi/k$ above since the propagator of the gauge bosons and the scalar field is different. This turns out to be a convenient normalization in perturbation theory. We remark that the identification (4.9) can also be written as

$$\dot{a}_{ab} + i\dot{b}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{I=1}^4 (\sigma^I)_{ab}\dot{z}^I \tag{4.10}$$

and our proposal for the Wilson loop operator that is dual to a string with the boundary condition (4.5) is

$$W = \frac{1}{N}\mathrm{Tr}P\exp\left[\oint_C d\tau\left(iA_\mu\dot{x}^\mu + \frac{2\pi}{k}\sum_{I=1}^{4}\dot{z}^I\bar{R}^I + \dot{\bar{z}}^I R^I\right)\right]. \tag{4.11}$$

Here R^I is the composite scalar $R^I := (\mathcal{A}^I + i\mathcal{B}^I)/\sqrt{2}$ where $\mathcal{A}^I := A_a(\sigma^I)_{ab}\bar{A}_b$, $\mathcal{B}^I := \bar{B}_a(\sigma^I)_{ab}B_b$.

By doing a perturbative computation as in, e.g. [35 − 37], one can show that the Wilson loop is in general linear divergent:

$$\sim \frac{N^2}{k^2\epsilon}\int d\tau_1(\dot{x}(\tau_1)^2 - \dot{y}(\tau_1)^2). \tag{4.12}$$

Therefore the divergence cancels if the loop constraint

$$\dot{x}^2 = \dot{y}^2 \tag{4.13}$$

is satisfied. The fact that we obtain precisely the same constraint as obtained from the Hamilton-Jacobi analysis provides some support that the ansatz (4.11) correctly encodes the boundary conditions of the dual open string. However this cannot be correct due to a mismatch. In fact, a half BPS string configuration which is localized at a point in CP^3 has been considered in [35 − 37]. One can show that there is no choice of $\dot{z}^I$ to make (4.11) half BPS. Even worse, it is easy to show, for the ansatz (4.1) which is coupled to a bilinear of scalars, there is no choice of the Hermitian matrix M so that there is 1/2 unbroken supersymmetry. Therefore the correct Wilson loop operator that is dual to localized string must be more complicated. The understanding of this will be very interesting.

Acknowledgments

We would like to thank Jaume Gomis, Asad Naqvi and Douglas Smith for stimulating discussions and useful comments. The research of CSC is partially supported by EPSRC and STFC. The work of DG is supported by an EPSRC studentship.

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